# R- ANNIHILATOR-HOLLOW AND R- ANNIHILATOR LIFTING MODULES

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***ABSTRACT:*** *Let M be a unitary left R- module on associative ring with identity R . A submodule K of M is called R-ann-small if K+T=M, where T is a submodule of M, implies that ann(T)=0, where ann(T) indicates annihilator of T in R . In this paper we introduce the concepts R-annihilator-hollow modules, R-annihilator-lifting modules and R-annihilator–amply supplemented modules. We give many properties related with this type of modules.*

**Keywords:** hollow module, lifting module, Small submodule, R- ann-hollow and R- ann-lifting modules

## INTRODUCTION

Let M be a unitary left R- module on associative ring with identity R . M is called a hollow module if every proper submodule is small in M, where a submodule N of R-module M is called small in M ($N \ll M$) if $N + K \neq M$ for each proper submodule K of M. A proper submodule N of R-module M is called an essential in M ($N \leq^e M$) if for every

nonzero submodule K of M then $N \cap K \neq 0$ [1].

In [2] there was given the concept of R-ann-small submodules . A submodule N of a R- module M is called R- ann- small if N+T=M, T a submodule of M , implies that $ann_R$ (T) =0 , where $ann_R$ (T) ={r R: r.T=0} .In this paper we give the concept of R- annihilator -hollow and R-annihilator -lifting modules , Let U and V be submodules of an R-module M. We say that M is R-annihilator-hollow module if every proper submodule of M is R-a-small in M. And R-module M is called R- annihilator –lifting if for any submodule N of M there exist submodules K , $K'$ of M such that $M = K \oplus K'$ with $K \leq N$ and $N \cap K'$ is R-annihilator -small in N and we introduce the concept R-annihilator –amply supplemented .Some properties of these modules are considered and give some characterizations for such modules.

### 1. R- annihilator-hollow module

Now we recall the definition and properties of R- ann -small submodule and introduce the definition of R-annihilator-hollow module .

**Definition 1.1:[2]** A submodule N of a module M is R-annihilator-small in M(R-a-small) if N+X=M , X a submodule of M ,implies that annX=0 , we write $N \ll^a M$ in this case .

**Remarks1.2**: [2]

**1-** Let A and B be submodules of M such that $A \leq B$ . if $A \ll^a B$, then $A \ll^a M$ .

**2-** Let A and B be submodules of a module M such that $A \leq B$. If $B \ll^a M$, Then $A \ll^a M$.

**3-** Let $f:M \rightarrow N$ be an epimorphism . If $H \ll^a N$, then $f^{-1}(H) \ll^a M$.

4- Let $M = M_1 \oplus M_2$. If $N_1$and $N_2$ are R-a-small submodules of $M_1$, $M_2$ respectively thus $N_1 \oplus N_2$ is an R-a-small submodule of $M_1 \oplus M_2$.

Now we can prove the following.

**Proposition1.3:** Let $M = D_1 \oplus D_2$ such that ann $D_2 \leq^e R$ , and $A \leq D_1$ . If $A \ll^a M$, Then $A \ll^a D_1$.

Proof: Let $M = D_1 \oplus D_2$, and $A \leq D_1$ to show that $A \ll^a D_1$ set $M_1$ =A+B, then M= A+B +$D_2$, $A \ll^a M$ then ann(B+$D_2$)=0, ann(B+ $D_2$)= ann(B)$\cap$ann($D_2$) but ann $D_2 \leq^e R$ thus ann(B)=0 and $A \ll^a D_1$.

**Corollary1.4:** Let $M = D_1 \oplus D_2$ such that ann $D_1 \leq^e R$ and ann $D_2 \leq^e R$ , and let $N \leq M$ ,$N = N_1 \oplus N_2$. If $N \ll^a M$ then $N_1 \ll^a D_1$ and $N_2 \ll^a D_2$.

**Definition 1.5:** A nontrivial module M is called R-annihilator -hollow(R- a-hollow) if every proper submodule of M is R-a-small in M.

**Examples and Remarks 1.6 :**

**(1)** Z as Z- module is R- a-hollow module but it is not hollow.

**(2)** $Z_6$ and $Z_4$ as Z- module are not R- a- hollow modules.

**(3)** Simple modules are hollow module but not R- a-hollow modules.

**(4)** If M a torsion free module on integeral domain R,then M is R- a- hollow.

**(5)** If M be a faithful R-module , then R- a-hollow and hollow are equivalent.

**Proposition 1.7:** Let M be a faithful module on R, and ann(N) is essential for every N be a submodule of M ,then M is R- a-hollow module.

**Proof:** Let D be submodule of M ,then by (prop.2. 1.13[2]), D is R-a-small submodule.

The epimorphic image of ;R-a-hollow module need not be R- a- hollow as the following example shows:-

Consider Z and $Z_4$as Z – modules and $\pi : Z \rightarrow Z_4$ let be the natural epimorphism . Z as Z- module is R- a-hollow module ,$\{0\} \ll^a Z$ . But $\pi(\{0\})$=0 is not Z-a-small in $Z_4$ , where

$Z_4 = 0 + Z_4$ and ann $Z_4$ =4Z $\neq 0$.

**Proposition1-8:-**Let M and N be two modules on R , and $f:M \rightarrow N$ be an epimorphism .If N is R-a-hollow module then M is R-a-hollow module.

**Proof:-**Let K be submodule of M ,then f(K) is submodule of N and since N is R-a-hollow module , f(K) is R-a-small submodule, then $f^{-1}(f(k)) \ll^a M$ by (Remark1.2(3)), $f^{-1}(f(k))$=K+kerf , $K \leq$ K+kerf then by(Remark 1.2(1)), K is R-a-small submodule of M.

**Corollary 1.9:** Let M be a module on R , D be submodule of M . If M/D is R- a-hollow module then M is R- a-hollow module.

A submodule N of R-module M is called fully invariant submodule of M if $f(N) \subseteq N$, for every $f \in Hom(M,M)$ .A module M is called duo module if every submodule of M is fully invariant[3].

**Proposition 1.10:** Let $M = D_1 \oplus D_2$ be duo module .If $D_1$ and $D_2$ are R- a-hollow modules, then M is R- a-hollow module.

**Proof:** Let $D_1$ and $D_2$ be R- a-hollow modules, and $N_1 \oplus N_2$ be a proper submodule of $D_1 \oplus D_2$

$N_1 \leq D_1$ and $N_2 \leq D_2$, then $N_1$ and $N_2$ are R-a-small submodules of $D_1$, $D_2$ respectivly thus by(Remark 1-2(4)) $N_1 \oplus N_2$ is R-a-small submodule of M.

**Corollary 1.11:** Let $M= D_1 \oplus D_2$ be an R- module such that $R= ann(D_1)+ann(D_2)$. If $D_1$ and $D_2$ are be R- a-hollow , then so is M.

A ring R is called R-a-hollow, if R is R-a-hollow R-module.

A module M is multiplication, if for every submodule F of M there exists an ideal I of R such that F = IM = (F:M)M [4].

**Proposition 1.12:** Let M be a multiplication R-module. If M is R- a-hollow then R is a R- a-hollow ring.

**Proof:** Suppose that M is R- a-hollow. Let I be an ideal of R. Then IM is a submodule of M and hence IM is R-a-small ([2], Prop.2.1.17). Then I is R-a-small ideal of R and hence R is R-a-hollow.

**2. R- annihilator-lifting module :**

M is lifting module if for any submodule N of M there exist submodules L, K of M such that $M=L\oplus K$ with $L\leq N$ and $N\cap K \ll N$ (equivalently $N\cap K\ll M$) [5]. In this section we introduce the notion of R-annihilator-lifting module (R- a-lifting ) and discus some properties of this kind of modules.

**Definition 2.1:** M is called R-annihilator-lifting (R- a-lifting) if for any submodule N of M there exist submodules L, K of M such that $M=L\oplus K$ with $L\leq N$ and $N\cap K \ll^a N$.

By using Remark (1.2) we get the following remark.

**Remark 2.2:** A module M is R- a-lifting if and only if for any submodule N of M there exists a submodule L of M such that $M=L\oplus K$ and $N\cap K \ll^a M$.

We shall call a ring R, R- a-lifting if R is a R-a-lifting as an R-module. The following proposition gives a characterization of R- a-lifting modules.

**Proposition 2.3:** Let M be a R- a-lifting then every submodule N of M, can be written as $N=A\oplus B$ where A is direct summand of M and $B\ll^a M$.

**Proof:** is trivial**.**

**Remark 2.4:** Every R- a-hollow module is R- a-lifting module.

**Proof:** Let N be submodule of M if $N\neq M$, R- a-hollow module then $N\ll^a M$,

$M= (0) \oplus M$ with $(0)\leq N$ and $N\cap M \ll^a M$.

**Proposition 2.5:** Let $M=H_1\oplus H_2$ be duo module. If $H_1$ and $H_2$ are R-a-lifting modules, then M is R-a-lifting module.

**Proof:** Let $H_1$ and $H_2$ are R-a-lifting modules, let N submodule of M, then $N= (N\cap H_1)\oplus(N\cap H_2)$ .For each $i\in\{1,2\}$,there exists a direct summund $D_i$ of $H_i$ ,such that $H_i= D_i \oplus L_i$ with $D_i \leq N\cap Hi$ and $N\cap L_i \ll^a L_i$ then, $M=(D_1\oplus L_1)\oplus(D_2\oplus L_2)=(D_1\oplus D_2)\oplus(L_1\oplus L_2)$, we have $(D_1\oplus D_2) \leq N$, and $N\cap(L_1\oplus L_2)\ll^a (L_1\oplus L_2)$ by (Remark 1.2) then M is R- a-lifting module.

**Corollary2-6:** Let $M=H_1\oplus H_2$ be a module such that $R=ann(H_1)+ann(H_2)$. If $H_1$ and $H_2$ are R-a-lifting modules, then so is M.

**Theorem 2.7:** Let M be a multiplication R -module. if M is R- a-lifting module, then R is R- a-lifting ring.

**Proof:** Assume that M is R- a-lifting module . Let I be an ideal of R. Then N = IM is a submodule of M, hence there exist submodules K and K' of M with $K \subseteq N$, $M = K \oplus K'$ and $(N \cap K') \ll^a M$ (by Def. 2.1). But M is a multiplication R module, so there are ideals J and J' of R such that K = JM and K' = J'M. We get $J \subseteq I$ (since $K \subseteq N$). We have $M = K\oplus K' = JM\oplus J'M = (J\oplus J')M$ implies that $R = J\oplus J'$. Now, $N \cap K' = (IM \cap J'M) \ll^a M$ and since $(J\cap J')M \subseteq IM \cap J'M$ it follows that $(J\cap J')M \ll^a M$ (Remark 1.2) and according to ([2] cor.2.1.18), we get $[(J\cap J')M:M] \ll^a R$. But $[(J\cap J')M:M] = I \cap J'$, therefore $(I\cap J')\ll^a R$ then R is an R-a-lifting ring.

**3. R-ann-amply supplemented module:**

In [6] Al-Hurmuzy and Al-Bahrany**,** introduce the concept of R-annihilator supplemented (R-a-supplemented) module. In this section we introduce the concept of R- annihilator amply supplemented (R-a-amply supplemented) module. We also give some basic properties of this class of modules.

D**efinition 3.1 [6]:** Let V and U be submodules of an R-module M. V is R-annihilator-supplement (R-a-supplement) of U in M if M=U+V and whenever $Y\leq V$ and M=U+Y, then ann Y=0.

Let M be R-module. M is R-annihilator–supplemented (R-a-supplemented) module if every proper submodule of M has R-a-supplement.

**Proposition 3.2 [6]:** Let U and V be submodules of R-module M. Then V is R-a-supplement of U if and only if M=U+V and $U\cap V \ll^a V$.

Now we introduce the following concept.

**Definition 3.3:** Let M be an R-module. M is R-annihilator–amply supplemented (R-a-amply supplemented) module if for any submodules A, B of M with M=A+B there exists an R-a-supplement K of A such that $K \leq B$.

**Proposition 3.4:** Let M and N be R-modules and let $f: M\rightarrow N$ be an epimorphism if N is R-a-amply supplemented module, then M is R-a-amply supplemented module.

**Proof:** Let A, B be submodules of M with M=A+B, then N=f(A)+f(B)since N is R-a-amply supplemented module, there exists a submodule K of N such that N= f(A)+K , $f(A) \cap K \ll^a K \leq f(B)$. $M= f^{-1}( N)= )$, $f^{-1}(f(A)+ K)=A+ \ker f + f^{-1}( K) )=A + f^{-1}( K)$. Since $f(A) \cap K \ll^a K$ then by (Remark1-2) , $f^{-1} (f(A) \cap K ) \ll^a f^{-1}( K)$ but $f^{-1} (f(A) \cap K )=(A+ \ker f )\cap f^{-1}( K)= \ker f +(A\cap f^{-1}( K)) \ll^a f^{-1}( K)$ by (Remark1-2) $A\cap f^{-1}( K) \ll^a f^{-1}(K)$ so $f^{-1}(K)$ is R-a- supplement of A and $K \leq f(B)$ then $f^{-1}(K) \leq B$ thus M is R-a-amply supplemented .

**Proposition 3.5:** Let M be an R-modules. If every submodule of M is an R-a-supplemented module, then M is an R-a-amply supplemented module.

**Proof:** Let A, B be submodules of M with M=A+B, by assumption there is $H\leq B$ such that ,$B=(A \cap B)+H$ and $(A \cap B) \cap H =A \cap H \ll^a H$. Thus $B=(A \cap B)+H \leq H+A$ , since $M=A+B\leq H+A$ then M=A+H.

**Corollary 3.6:** For any ring R, the following are equivalent.

1-All modules are R-a-amply supplemented modules.

2-All modules on R are R-a- supplemented modules.

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